

# Dirac monopole with Feynman brackets

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## Abstract

We introduce the magnetic angular momentum as a consequence of the structure of the  $\mathfrak{so}(3)$  Lie algebra defined by the Feynman brackets. The Poincaré momentum and Dirac magnetic monopole appears as a direct result of this framework.

## I. INTRODUCTION

In 1990, Dyson [1] published a proof due to Feynman of the Maxwell equations, assuming only commutation relations between position and velocity. In this article we don't use the commutation relations explicitly. In fact what we call a commutation law is a structure of algebra between position and velocity called in this letter Feynman's brackets. With this minimal assumption Feynman never supposed the existence of an Hamiltonian or Lagrangian

formalism and didn't need the not gauge invariant momentum. Tanimura [2] extended Feynman's derivation to the case of the relativistic particle.

In this letter one concentrates only on the following point: the study of a nonrelativistic particle using Feynman brackets. We show that Poincare's magnetic angular momentum is the consequence of the structure of the  $\mathfrak{so}(3)$  Lie algebra defined by Feynman's brackets.

## II. FEYNMAN BRACKETS

Assume a particle of mass  $m$  moving in a three dimensional Euclidean space with position:  $x_i(t)$  ( $i = 1, 2, 3$ ) depending on time. As Feynman we consider a non associative internal structure (Feynman brackets) between the position and the velocity. The starting point is the bracket between the various components of the coordinate:

$$[x_i, x_j] = 0 \tag{1}$$

We suppose that the brackets have the same properties than in Tanimura's article [2], that is:

$$[A, B] = -[B, A] \tag{2}$$

$$[A, BC] = [A, B]C + [A, C]B \tag{3}$$

$$\frac{d}{dt}[A, B] = [\dot{A}, B] + [A, \dot{B}] \tag{4}$$

where the arguments  $A$ ,  $B$  and  $C$  are the positions or the velocities.

The following Jacobi identity between positions is also trivially satisfied:

$$[x_i, [x_j, x_k]] + [x_j, [x_k, x_i]] + [x_k, [x_i, x_j]] = 0 \tag{5}$$

In addition we will need also a "Jacobi identity" mixing position and velocity such that:

$$[\dot{x}_i, [\dot{x}_j, x_k]] + [\dot{x}_j, [x_k, \dot{x}_i]] + [x_k, [\dot{x}_i, \dot{x}_j]] = 0 \tag{6}$$

Deriving (1) gives:

$$[\dot{x}_i, x_j] + [x_i, \dot{x}_j] = 0 \quad (7)$$

This implies:

$$[x_i, \dot{x}_j] = g_{ij}(x_k), \quad (8)$$

where  $g_{ij}(x_k)$  is a symmetric tensor. We consider here only the case where:

$$g_{ij} = \frac{\delta_{ij}}{m} \quad (9)$$

this gives the following relations:

$$[x_i, f(x_j)] = 0 \quad (10)$$

$$[x_i, f(x_j, \dot{x}_j)] = \frac{1}{m} \frac{\partial f(\dot{x}_j)}{\partial \dot{x}_i} \quad (11)$$

$$[\dot{x}_i, f(x_j)] = -\frac{1}{m} \frac{\partial f(x_j)}{\partial x_i} \quad (12)$$

### III. ANGULAR MOMENTUM

Suppose first the following relation:

$$[\dot{x}_i, \dot{x}_j] = 0 \quad (13)$$

which permits to say that the force law is velocity independent:

$$\ddot{x}_i = \ddot{x}_i(x_j) \quad (14)$$

By definition the orbital angular momentum is:

$$L_i = m \varepsilon_{ijk} x_j \dot{x}_k \quad (15)$$

which satisfies the standard  $\mathfrak{so}(3)$  Lie algebra for Feynman's brackets:

$$[L_i, L_j] = \varepsilon_{ijk} L_k \quad (16)$$

The transformation law of the position and velocity under this symmetry is:

$$[x_i, L_j] = \varepsilon_{ijk} x_k \quad (17)$$

$$[\dot{x}_i, L_j] = \varepsilon_{ijk} \dot{x}_k \quad (18)$$

We consider as Feynman [1], the case with a "gauge curvature":

$$[\dot{x}_i, \dot{x}_j] = \frac{\alpha}{m^2} F_{ij} \quad (19)$$

where  $F$  must be an antisymmetric tensor (electromagnetic tensor for our example) and  $\alpha$  a constant. The goal of our work is to see what happens if we keep the structure of the Lie algebra of the angular momentum and the transformation law of the position and velocity.

Using (6) we get the relations:

$$\begin{aligned} \alpha \frac{\partial F_{jk}}{\partial \dot{x}_i} &= -m^2 [x_i, [\dot{x}_j, \dot{x}_k]] \\ &= -m^2 [\dot{x}_j, [x_i, \dot{x}_k]] + [\dot{x}_k, [\dot{x}_j, x_i]] = 0 \end{aligned} \quad (20)$$

then the electromagnetic tensor is independent of the velocity:

$$F_{jk} = F_{jk}(x_i) \quad (21)$$

By deriving (8) we have:

$$[x_i, \ddot{x}_j] = -[\dot{x}_i, \dot{x}_j] = -\frac{\alpha F_{ij}}{m^2} \quad (22)$$

then:

$$m \frac{\partial \ddot{x}_j}{\partial \dot{x}_i} = \alpha F_{ji}(x_k) \quad (23)$$

or:

$$m \ddot{x}_i = \alpha (E_i(x_k) + F_{ij}(x_k) \dot{x}_j) \quad (24)$$

We get the " Lorentz force's law", where the electric field appears as a constant of integration (this is not the case for the relativistic problem, see [2]). Now the force law is velocity dependent:

$$\ddot{x}_i = \ddot{x}_i(x_j, \dot{x}_j) \quad (25)$$

For the case (19), the equations (16), (17) and (18) become :

$$[x_i, L_j] = \varepsilon_{ijk} x_k \quad (26)$$

$$[\dot{x}_i, L_j] = \varepsilon_{ijk} \dot{x}_k + \alpha \varepsilon_{jkl} x_k \frac{F_{il}}{m} \quad (27)$$

$$[L_i, L_j] = \varepsilon_{ijk} L_k + \alpha \varepsilon_{ikl} \varepsilon_{jms} x_k x_m F_{ls} \quad (28)$$

Introducing the magnetic field we write  $F$  in the following form:

$$F_{ij} = \varepsilon_{ijk} B_k, \quad (29)$$

We get then the new relations:

$$[\dot{x}_i, L_j] = \varepsilon_{ijk} \dot{x}_k + \frac{\alpha}{m} \{x_i B - \delta_{ij} (\vec{r} \cdot \vec{B})\} \quad (30)$$

$$[L_i, L_j] = \varepsilon_{ijk} \{L_k + \alpha x_k (\vec{r} \cdot \vec{B})\} \quad (31)$$

To keep the standard relations we introduce a generalized angular momentum:

$$\mathcal{L}_\gamma = \mathcal{L}_\gamma + \mathcal{M}_\gamma \quad (32)$$

We call  $M_i$  the magnetic angular momentum because it depends on the field  $\vec{B}$ . It has no connection with the spin of the particle, which can be introduced by looking at the spinorial representations of the  $\mathfrak{so}(3)$  algebra. Now we impose for the  $\{\alpha_j\}$ 's the following commutation relations:

$$[\dot{x}_i, \mathcal{L}_j] = \varepsilon_{ijl} \mathcal{L}_l \quad (33)$$

$$[\dot{x}_i, \mathcal{L}_\parallel] = \varepsilon_{\parallel} \dot{\mathcal{S}}_\parallel \quad (34)$$

$$[\mathcal{L}_\parallel, \mathcal{L}_\parallel] = \varepsilon_{\parallel} \mathcal{L}_\parallel \quad (35)$$

This first relation gives:

$$M_i = M_i(x_j) \quad (36)$$

and the second:

$$[\dot{x}_i, M_j] = \frac{\alpha}{m} [\delta_{ij}(\vec{r} \cdot \vec{B}) - x_i B_j] \quad (37)$$

If we replace it in (35) we deduce:

$$M_i = -\alpha(\vec{r} \cdot \vec{B})x_i \quad (38)$$

Putting this result in (34) gives the following equation of constraint for the field  $\vec{B}$  :

$$x_i B_j + x_j B_i = -x_j x_k \frac{\partial B_k}{\partial x_i} \quad (39)$$

One solution has the form of a radial vector field centered at the origin:

$$\vec{B} = \beta \frac{\vec{r}}{r^3} \quad (40)$$

The generalized angular momentum then becomes:

$$\vec{\mathcal{L}} = m(\vec{r} \wedge \dot{\vec{r}}) - \alpha(\vec{r} \cdot \vec{B}) \vec{r} \quad (41)$$

We can check the conservation of the total angular momentum:

$$\frac{d\vec{\mathcal{L}}}{dt} = m(\dot{\vec{r}} \wedge \ddot{\vec{r}}) - \alpha\{\dot{\vec{r}} \wedge (\dot{\vec{r}} \wedge \vec{B})\} = 0 \quad (42)$$

because the particle satisfies the usual equation of motion:

$$m \frac{d^2 \vec{r}}{dt^2} = \alpha(\dot{\vec{r}} \wedge \vec{B}) \quad (43)$$

If we choose:  $\alpha = q$  and  $\beta = g$ , where  $q$  and  $g$  are the electric and magnetic charges, we obtain as a the special case the Poincaré [3] magnetic angular momentum:

$$\vec{M} = -\frac{qg}{4\pi} \frac{\vec{r}}{r} \quad (44)$$

and the Dirac [4] magnetic monopole:

$$\vec{B} = \frac{g}{4\pi} \frac{\vec{r}}{r^3} \quad (45)$$

In addition we find that for the Dirac monopole the source of the field is localized at the origin:

$$\text{div} \vec{B} = [\dot{x}_i, [\dot{x}_j, \dot{x}_k]] + [\dot{x}_j, [\dot{x}_k, \dot{x}_i]] + [\dot{x}_k, [\dot{x}_i, \dot{x}_j]] = \frac{g}{4\pi} [\dot{x}_i, \frac{x_i}{r^3}] = g\delta(\vec{r}) \quad (46)$$

We see that in the construction of the Feynman's brackets algebra the fact that we didn't impose the Jacobi identity between the velocities is a necessary condition to obtain a monopole solution.

In summary, we used the Feynman's algebra between position and velocity to compute the algebra of the angular momentum of a non relativistic particle in a electromagnetic field. The Dirac monopole and magnetic angular momentum is a direct consequence of the conservation of the form of the standard  $\text{sO}(3)$  Lie algebra.

#### IV. CASIMIR OPERATOR

In the same spirit, it is interesting to introduce  $L^2$ , the Casimir operator of  $\text{sO}(3)$  Lie algebra. Again we want to keep the same commutation relations in the two cases corresponding to zero and non zero curvature.

In the first case, we easily see that:

$$[x_i, L^2] = 2(\vec{L} \wedge \vec{r})_i \quad (47)$$

$$[\dot{x}_i, L^2] = 2(\vec{L} \wedge \dot{\vec{r}})_i \quad (48)$$

$$[L_i, L^2] = 0 \quad (49)$$

and in presence of a curvature:

$$[x_i, L^2] = 2(\vec{L} \wedge \vec{r})_i \quad (50)$$

$$[\dot{x}_i, L^2] = 2[(\vec{L} \wedge \dot{\vec{r}})_i + \alpha(\vec{L} \wedge \vec{r})_l F_{il}] \quad (51)$$

$$[L_i, L^2] = 2\alpha(\vec{L} \wedge \vec{r})_i(\vec{r} \cdot \vec{B}) \quad (52)$$

then we want:

$$[x_i, \mathcal{L}^\epsilon] = \epsilon(\vec{\mathcal{L}} \wedge \vec{\nabla})_i \quad (53)$$

$$[\dot{x}_i, \mathcal{L}^\epsilon] = \epsilon(\vec{\mathcal{L}} \wedge \dot{\vec{\nabla}})_i \quad (54)$$

$$[\mathcal{L}, \mathcal{L}^\epsilon] = \epsilon \quad (55)$$

and we can deduce:

$$[x_i, M^2] = 2(\vec{M} \wedge \vec{r})_i \quad (56)$$

$$[\dot{x}_i, M^2] = 2[(\vec{M} \wedge \dot{\vec{r}})_i - \alpha(\vec{L} \wedge \vec{r})_l F_{il}] \quad (57)$$

$$2\alpha(\vec{L} \wedge \vec{r})_i(\vec{r} \cdot \vec{B}) + [L_i, M^2] + [M_i, L^2] = 0 \quad (58)$$

The last equation becomes after a straightforward computation:

$$(\vec{M} \wedge \vec{r})(\vec{L} \wedge \dot{\vec{r}}) - (\vec{L} \wedge \vec{r})(\vec{M} \wedge \dot{\vec{r}}) - (\vec{M} \wedge \dot{\vec{r}})(\vec{L} \wedge \vec{r}) + (\vec{M} \wedge \vec{r})(\vec{L} \wedge \dot{\vec{r}}) = 0 \quad (59)$$

We can check that this equation of constraint is in particular satisfied for the Poincaré angular momentum.



## V. CONCLUSION

We find that the structure of Feynman's brackets (without an Hamiltonian or Lagrangian), illuminates the connections between the spaces with gauge curvature, the  $\mathfrak{so}(3)$  Lie algebra and the existence of the Poincaré magnetic angular momentum. It seems that more than the phase space formalism, the Feynman's one is a good approach of the mechanics in a space with gauge symmetry, because it avoids the introduction of the not gauge invariant momentum. Further, other applications of this method, for example, the case of the Minkowski space with Lorentz Lie algebra, will be consider in the future.

## REFERENCES

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